On the Riccati Partial Differential Equation for Nonlinear Bolza and Lagrange Problems

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Abstract

In this paper, we study the solution of certain optimal control problems by the use of an off-line Riccati Partial Differential Equation which is shown to be equivalent to the Hamilton-Jacobi-Bellman (HJB) equation, when the value function is sufficiently smooth. We develop a geometric existence theory for classical, weak and generalized solutions of the Riccati Partial Differential Equation. For finite and infinite time horizon problems, we also investigate the relationship of such solutions to optimal control laws. Indeed, as corollaries of certain of our existence results, we obtain smoothness results for the value function of the corresponding optimal control problem.

1 Introduction

Consider a control system having the form

$$\dot{x} = f(x) + g(x)u(t) \tag{1.1}$$

where $x \in \mathbb{R}^n$, and where for each t, $u(t) \in \mathbb{R}^m$. We assume that the vector fields f, g_i are C^r in x, $r \geq 1$ and that the $u_i(t)$ are piecewise continuous functions. In particular, for each pair $(x(0),\ u(t))$, the system (1.1) has a unique solution for $t << \infty$. Furthermore, we assume that $f(0) = 0$. For the system (1.1) we shall consider the problem of minimizing the cost functional

$$J^T(x(0), u) = \int_0^T L(x, u)dt + Q(x(T)) \tag{1.2}$$

for both the cases $T < \infty$, and for $T = \infty$ and $Q(x) \equiv 0$.

Our initial assumptions concerning (1.2) are that $Q(x)$ is C^{q+1}, $q \geq 1$ and that $L(x, u)$ is C^{s+1}, $s \geq 1$, and satisfies:

H1: for each fixed x, $\frac{\partial L}{\partial u}(x, \cdot)$ is a diffeomorphism;

H2: for each fixed x, $L(x, u)$ has a minimum at $u = 0$, and $L(0, 0) = 0$.

In particular, we note that $\frac{\partial^2 L}{\partial u^2}(x,u) > 0$ for all (x,u) and therefore that (H1)–(H2) imply that $L(x,u)$ is strictly convex in u.

H3: 0 is a critical point of Q, and $Q(0) = 0$.

We consider the Hamiltonian

$$H(x,p,u) = \langle p, f(x) + g(x)u \rangle - L(x,u)$$

and note that, by (H1), for each (x,p) there is a unique $u_*(x,p)$, C^k in (x,p), which satisfies

$$0 = \frac{\partial H}{\partial u}|_{u=u_*} = \langle p, g(x) \rangle - \frac{\partial L}{\partial u}(x, u_*)$$

Moreover, in the light of (H2), for each fixed pair (x,p), the value $u_*(x,p)$ in fact maximizes $H(x,p,u)$ Defining a Hamiltonian function, $H_*(x,p)$, via $H_*(x,p) = H(x,p,u_*(x,p))$ we note that according to the Maximum Principle, for the Bolza Problem every extremal control $u_*(t)$ for any initial condition $x(0)$ gives rise to a trajectory, or canonical pair, $(x(t), p(t))$ satisfying

$$\dot{x}(t) = \frac{\partial H_*}{\partial p} \tag{1.3}$$

$$\dot{p}(t) = -\frac{\partial H_*}{\partial x} \tag{1.4}$$

and

$$p(T) = -\nabla Q(x(T)). \tag{1.5}$$

We shall first impose a further simplifying asumption on (1.1)–(1.2).

H4: The canonical system (1.3)–(1.4) is complete.

Now consider the closed, connected C^k submanifold of $\mathbb{R}^{2n}$ defined via

$$M_T = \{(x,p) : p = -\nabla Q(x)\} \tag{1.6}$$

M_T is of course the submanifold of terminal constraints given by the transversality conditions. We note that for $t \in [0,T]$

$$M_t = \Phi_{t-T}(M_T) \tag{1.7}$$

is a closed, connected Lagrangian C^k submanifold of $\mathbb{R}^{2n}$ consisting of those pairs $(x(t), p(t))$ which satisfy (1.3)–(1.4), with initial time $t_0 = t$. In particular, for $s \in [t,T]$

$$u(s) = u_*(x(s), p(s))$$

is an extremal control for the Bolza problem (1.1)–(1.2) with initial time $t_0 = t$. Moreover, to say

$$M_s = \text{graph}(-\pi(x,s)), \quad t \le s \le T$$

is to say that all extremal controls for $t \le s \le T$ can be given in feedback form

$$u(s) = u_*(x(s), -\pi(x(s), s)), \quad t \le s \le T.$$

We first derive conditions on $\pi(x,t)$ so that the geometric condition (1.8) will be satisfied.

Theorem 1.1. *Necessary and sufficient conditions for the existence of a C^ℓ function*

$$\pi(x,t), \quad 1 \le \ell \le k,$$

such that

$$M_t = graph(-\pi(x,t))$$

for $t \in [t_0, T]$, $x \in \mathbb{R}^n$ is that $\pi(x,t)$ satisfy the following "Riccati" partial differential equation, for $(x,t) \epsilon \mathbb{R}^n \times (t_0, T)$

$$\frac{\partial \pi}{\partial t} = \frac{\partial H_*}{\partial x}(x, -\pi) - \frac{\partial \pi}{\partial x}\frac{\partial H_*}{\partial p}(x, -\pi) \tag{1.8}$$

$$\pi(0,t) = 0, \quad \pi(x,T) = \nabla Q(x) \tag{1.9}$$

In particular, the Riccati partial differential equation has a C^k solution if, and only if, it has a C^1 solution.

We prove solvability of the Riccati PDE with a C^ℓ function $\pi(x,t)$ is equivalent to solvability of the Hamilton-Jacobi-Bellman equation with a $C^{\ell+1}$ function $V(x,t)$ in C^{k+1} if, and only if, it is C^2.

If (M_t) is always the graph of a smooth function we say (M_t) is a strong solution of the Riccati equation. If (M_t) is the graph of a continuous function, we say there exists a weak solution. In general, the family (M_t) of Lagrangian submanifolds is referred to as a generalized solution.

In the hierarchy of classical, weak and generalized solutions of the Riccati PDE, we show that there is a corresponding hierarchy of regularity for the value function: To say the value function is C^2 is to say it is C^k, which is to say a classical solution of the Riccati PDE exists. To say the value function V is C^1 but not C^2 is to say V is C^k on an open dense subset and that a weak solution of the Riccati PDE exists. To say that V is not C^1 is to say the unique generalized solution of the Riccati PDE is not a weak solution but is instead multi-valued.

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References

[1] C.I. Byrnes, "Some Partial Differential Equations Arising in Nonlinear Control," *Computation and Control, II*, Birkhäuser-Boston, 1991.

[2] C.I. Byrnes, "New Methods for Nonlinear Optimal Control, " *Proc. of the 1st ECC*, 1991.

[3] C.I. Byrnes and H. Frankowska, "Unicité des solutions optimales et absence des chocs pour les équations d'Hamilton-Jacobi-Bellman et de Riccati," *C.R. Acad. Sci. Paris Sér I Math.*. 315 (1992), no. 4, 427-431.

[4] A.P. Willemstein, "Optimal Regulation of Nonlinear Systems on a Finite Interval, " *SIAM J. Contr. and Opt.* 15, 1977, 1050–1069.

[5] P. Brunovsky, "On optimal stabilization of nonlinear systems, " *Mathematical Theory of Control*, A.V. Balakrishnan and Lucien W. Neustadt, eds., Academic Press, New York and London, 1967.

[6] D. Lukes, "Optimal Regulation of Nonlinear Systems, " *SIAM J. Contr. and Opt.* 7, 1969, 75–100.

[7] C.I. Byrnes and A. Jhemi, "Shock waves for Riccati Partial Differential Equations for Optimal Control," *Systems, Models and Feedback: Theory and Applications*, Birkhauser-Boston, 1992 .